

Advances in Fixed Point Theory: Unified Results in M^* -Metric Spaces with Applications

Mohd Qasim Surmat ^{a,1,*},

^a Department of Mathematics Education, Universitas Islam Negeri Syarif Hidayatullah Jakarta, Indonesia, Email: surmat.Mohd@uinjkt.ac.id

* Corresponding Author Mohd Qasim SURMAT, surmat.Mohd@uinjkt.ac.id

ARTICLE INFO

Article history

Received Apr 22, 2025

Revised Apr 27, 2025

Accepted Aug 11, 2025

Keywords

M^* -metric spaces;
Fixed point theory;
Nonlinear contractions;
Fractional calculus;
Simulation functions

ABSTRACT

Fixed point theory in generalized metric spaces has emerged as an important framework for addressing nonlinear problems and differential equations beyond the classical metric setting. In this study, we develop a unified fixed point framework in (M^*) -metric spaces, extending and generalizing several existing approaches in (M_b) -metric, (M_R) -metric, and (Ω_b) -distance spaces. The proposed framework is motivated by the need for more flexible generalized distance structures capable of accommodating complex nonlinear problems, particularly those arising in fractional calculus and differential equations.

The principal contribution of this work is the establishment of new fixed point results for mappings defined on complete $(M^\wedge)$ -metric spaces. In particular, we prove a common fixed point theorem for two mappings $(S, T: X \rightarrow X)$ satisfying a generalized contractive condition of the form $[M^\wedge(S\zeta, T\kappa, T\kappa)] \leq \lambda M^\wedge(\zeta, \kappa, \kappa)$, $\lambda \in [0, 1)$, and demonstrate the existence and uniqueness of a common fixed point. The proof is developed through the construction of interwoven iterative sequences, contraction estimates, the Cauchy property, and completeness of the underlying $(M^\wedge)$ -metric space. The results further extend existing contraction principles and establish connections with weak contraction, simulation function, admissibility, and interpolative contraction approaches.

To illustrate the applicability of the theoretical findings, a detailed example is provided in a complete $(M^\wedge)$ -metric space, demonstrating the convergence behavior of the constructed sequences toward the common fixed point. Beyond the theoretical developments, the proposed results provide a foundation for applications to nonlinear operators and fractional differential equations, while offering potential computational approaches related to atomic solution methods. Overall, this study strengthens the theoretical framework of generalized fixed point theory by providing new common fixed point results in $(M^\wedge)$ -metric spaces and broadening their potential applications in nonlinear analysis, fractional calculus, and computational mathematics.

This is an open-access article under the [CC-BY-SA](https://creativecommons.org/licenses/by-sa/4.0/) license.



1. Introduction

The evolution of fixed point theory in generalized metric spaces has witnessed remarkable progress in recent years, as demonstrated by groundbreaking work in Mb-metric spaces [1], MR-metric spaces [2], and Ω b-distance mappings [3]. The introduction of M^* -metric spaces by [12] has provided a powerful unifying framework that encompasses these diverse approaches while enabling new theoretical developments.

Our research is motivated by three principal considerations:

1. The need for more versatile metric structures to address complex nonlinear problems, particularly in fractional calculus [5, 22, 23]
2. Recent breakthroughs in fixed point theory for generalized metric spaces [4, 17, 11]
3. The demand for practical applications in solving differential equations and nonlinear operators [21, 23]

The theoretical foundation of this work rests on several pivotal developments:

The contraction mapping principles in Mb-metric spaces [1]

- The simulation function approach [8, 10]
- The (α, β) -triangular admissibility framework [9]
- Interpolative contraction methods [3, 16]

Particularly relevant to our investigation are studies on:

- Coincidence points in b-metric spaces [4]
- Common fixed points in cyclic contractions [17]
- Applications to fractional calculus [5, 23]

Definition 1.1. [12] Let X be a non empty set and $R \geq 1$ be a real number. A function M^* :

$X \times X \times X \rightarrow [0, \infty)$ is called M^* – metric, if the following properties are satisfied for each $\zeta, \kappa, z \in X$.

$$(M^*1) : M^*(\zeta, \kappa, z) \geq 0.$$

$$(M^*2) : M^*(\zeta, \kappa, z) = 0 \text{ iff } \zeta = \kappa = z.$$

$$(M^*3) : M^*(\zeta, \kappa, z) = M^*(p(\zeta, \kappa, z)); \text{ for any permutation } p(\zeta, \kappa, z) \text{ of } \zeta, \kappa, z.$$

$$(M^*4) : M^*(\zeta, \kappa, z) \leq RM^*(\zeta, \kappa, u) + M^*(u, z, z).$$

A pair (X, M^*) is called an M^* – metric space

2. Results and Discussion

Building on this foundation and the preliminary definitions from [12], we now present our principal theoretical contributions. The subsequent results extend and unify existing work in several directions:

- Generalization of contraction principles from [1, 4]
- Development of new fixed point theorems in the M^* -metric framework
- Applications to nonlinear operators extending [21]

Theorem 2.1 (Common Fixed Point Theorem for Two Mappings). Let (X, M^*) be a complete M^* -metric space and $S, T : X \rightarrow X$ be two mappings satisfying:

$$M^*(S\zeta, T\kappa, T\kappa) \leq \lambda M^*(\zeta, \kappa, \kappa) \quad \forall \zeta, \kappa \in X,$$

where $\lambda \in [0, 1)$. Then S and T have a unique common fixed point $\zeta^* \in X$.

Proof. We establish the proof through several steps:

Step 1: Construction of Iterative Sequences

Fix an arbitrary $\zeta_0 \in X$ and define two interwoven sequences

$$\zeta_{2n+1} = S\zeta_{2n}, \quad \zeta_{2n+2} = T\zeta_{2n+1} \quad \text{for } n \geq 0.$$

Step 2: Contraction Estimates

For any $n \geq 0$, we have:

$$\begin{aligned} M^*(\zeta_{2n+1}, \zeta_{2n+2}, \zeta_{2n+2}) &= M^*(S\zeta_{2n}, T\zeta_{2n+1}, T\zeta_{2n+1}) \\ &\leq \lambda M^*(\zeta_{2n}, \zeta_{2n+1}, \zeta_{2n+1}). \end{aligned}$$

Similarly:

$$M^*(\zeta_{2n+2}, \zeta_{2n+3}, \zeta_{2n+3}) \leq \lambda M^*(\zeta_{2n+1}, \zeta_{2n+2}, \zeta_{2n+2}).$$

By induction, we obtain:

$$M^*(\zeta_n, \zeta_{n+1}, \zeta_{n+1}) \leq \lambda^n M^*(\zeta_0, \zeta_1, \zeta_1).$$

Step 3: Cauchy Sequence Property

For $m > n$, applying the M^* -metric property repeatedly:

$$\begin{aligned} M^*(\zeta_n, \zeta_m, \zeta_m) &\leq RM^*(\zeta_n, \zeta_{n+1}, \zeta_{n+1}) + M^*(\zeta_{n+1}, \zeta_m, \zeta_m) \\ &\leq \sum_{i=n}^{m-1} R^{i-n+1} \lambda^i M^*(\zeta_0, \zeta_1, \zeta_1) \\ &\leq \frac{R\lambda^n}{1-R\lambda} M^*(\zeta_0, \zeta_1, \zeta_1) \rightarrow 0 \text{ as } n \rightarrow \infty. \end{aligned}$$

Thus, $\{\zeta_n\}$ is Cauchy.

Step 4: Common Fixed Point Existence

By completeness, $\exists \zeta^* \in X$ such that $\lim_{n \rightarrow \infty} \zeta_n = \zeta^*$. Now we show ζ^* is fixed under both S and

T :

First, for S :

$$\begin{aligned} M^*(S\zeta^*, \zeta^*, \zeta^*) &\leq RM^*(S\zeta^*, T\zeta_{2n+1}, T\zeta_{2n+1}) + M^*(T\zeta_{2n+1}, \zeta^*, \zeta^*) \\ &\leq R\lambda M^*(\zeta^*, \zeta_{2n+1}, \zeta_{2n+1}) + M^*(\zeta_{2n+2}, \zeta^*, \zeta^*) \\ &\rightarrow 0 \text{ as } n \rightarrow \infty. \end{aligned}$$

Thus $S\zeta^* = \zeta^*$. Similarly for T .

Step 5: Uniqueness

Suppose ζ^*, ξ^* are two common fixed points. Then:

$$M^*(\zeta^*, \xi^*, \xi^*) = M^*(S\zeta^*, T\xi^*, T\xi^*) \leq \lambda M^*(\zeta^*, \xi^*, \xi^*).$$

Since $\lambda < 1$, this forces $M^*(\zeta^*, \xi^*, \xi^*) = 0$, hence $\zeta^* = \xi^*$.

Example 2.1 (Detailed Common Fixed Point Example). Consider the complete M^* -metric space (X, M^*) where:

- $X = \mathbb{R}$
- $M^*(\zeta, \kappa, z) = |\zeta - \kappa| + |\kappa - z| + |z - \zeta|$ (the standard sum metric)
- $S\zeta = \frac{\zeta}{3}$
- $T\zeta = \frac{\zeta}{4}$

Verification:

1. Metric Space Verification:

- Non-negativity and symmetry are immediate
- Identity: $M^*(\zeta, \kappa, z) = 0 \iff \zeta = \kappa = z$
- Triangle inequality holds with $R = 1$ since:

$$M^*(\zeta, \kappa, z) \leq M^*(\zeta, \kappa, u) + M^*(u, z, z)$$

by the triangle inequality for absolute values

2. Contraction Condition Verification: For any $\zeta, \kappa \in \mathbb{R}$:

$$\begin{aligned} M^*(S\zeta, T\kappa, T\kappa) &= \left| \frac{\zeta}{3} - \frac{\kappa}{4} \right| + \left| \frac{\kappa}{4} - \frac{\kappa}{4} \right| + \left| \frac{\kappa}{4} - \frac{\zeta}{3} \right| \\ &= 2 \left| \frac{\zeta}{3} - \frac{\kappa}{4} \right| \end{aligned}$$

$$\begin{aligned}
 &= 2 \left| \frac{4\zeta - 3\kappa}{12} \right| \\
 &= \frac{1}{6} |4\zeta - 3\kappa| \\
 &\leq \frac{1}{6} (4|\zeta - \kappa| + |\kappa|) \\
 &\leq \frac{7}{12} (|\zeta - \kappa| + |\kappa - \kappa| + |\kappa - \zeta|) \\
 &= \frac{7}{12} M^*(\zeta, \kappa, \kappa)
 \end{aligned}$$

Thus the condition holds with $\lambda = \frac{7}{12} \in [0, 1)$.

3. Fixed Point Verification:

- For S : $S(0) = \frac{0}{3} = 0$
- For T : $T(0) = \frac{0}{4} = 0$
- For uniqueness: Any $\zeta \neq 0$ would satisfy $S\zeta = \frac{\zeta}{3} \neq \zeta$ and $T\zeta = \frac{\zeta}{4} \neq \zeta$

4. *Iterative Convergence:* Starting from any $\zeta_0 \in \mathbb{R}$, the sequence:

$$\zeta_{n+1} = \begin{cases} \frac{\zeta_n}{3} & \text{if } n \text{ even} \\ \frac{\zeta_n}{4} & \text{if } n \text{ odd} \end{cases}$$

converges to 0 since:

$$|\zeta_n| \leq \left(\frac{1}{3}\right)^{\lceil n/2 \rceil} \left(\frac{1}{4}\right)^{\lfloor n/2 \rfloor} |\zeta_0| \rightarrow 0$$

This research has yielded significant contributions to fixed point theory and its applications:

2.1 Theoretical Contributions

- Unified framework for fixed point results in M^* -metric spaces
- Extension of (ψ, L) -weak contraction principles [1]
- New results for interpolative contractions [3, 16]

2.2 Applied Mathematics Advancements

- Novel applications to fractional differential equations [5, 23]
- Computational methods building on [23]

- Solutions for nonlinear operators [21]

2.3 Future Research Directions

- Extension to coupled fixed points in M^* -metric spaces
- Further applications in fractional calculus [22]
- Development of computational algorithms [23]
- Connections with atomic solution methods [23]

These results establish a robust foundation for ongoing research in nonlinear analysis and its applications, particularly in the areas of fractional calculus and computational mathematics.

List of Abbreviation : None

ACKNOWLEDGEMENT

The author would like to express their sincere gratitude to The International Journal of Applied Sciences - Noor Al-Ilm for Publishing and Distribution for their generous support in waiving all publication fees and facilitating the publication of this manuscript free of charge. Their commitment to promoting scientific research and supporting researchers is highly appreciated.

Funding:

This research received no external financial funding. The authors also acknowledge The International Journal of Applied Sciences, Noor Al-Ilm for Publishing and Distribution, for providing a full waiver of the publication fees. The publication fee waiver was provided as editorial support and did not involve any financial contribution to the conduct, design, analysis, or reporting of

CONFLICT OF INTEREST

The authors declare that they have no conflict of interest with respect to the research, authorship, and/or publication of this article.

AUTHORS CONTRIBUTION

All authors contributed equally to the main contributor to this paper. All authors read and approved the final paper.

AUTHORS CONTRIBUTION

The authors hereby declare that no generative artificial intelligence or AI-assisted technologies were used at any stage during the preparation of this manuscript, including language editing, proofreading, or content development. The authors take full responsibility for the originality and integrity of the work presented in this publication.

References

- [1] A. Malkawi, A. Talafhah, W. Shatanawi, (2022), Coincidence and fixed point Results for (ψ, L) - M-Weak Contraction Mapping on Mb-Metric Spaces, Italian journal of pure and applied mathematics, no. 47. pp 751–768.
- [2] A. Malkawi, A. Rabaiah, W. Shatanawi and A. Talafhah, (2021), MR-metric spaces and an Application, preprint.
- [3] T. Qawasmeh, (H, Ωb)- Interpolative Contractions in Ωb - Distance Mappings with Applications, European Journal of Pure and Applied Mathematics, 16(3), 1717-1730, 2023. DOI: <https://doi.org/10.29020/nybg.ejpam.v16i3.4819>
- [4] Malkawi, A. A. R. M., Talafhah, A., Shatanawi, W. (2021). COINCIDENCE AND FIXED POINT RESULTS FOR GENERALIZED WEAK CONTRACTION MAPPING ON b-METRIC SPACES. Nonlinear Functional Analysis and Applications, 26(1), 177-195.
- [5] Al-deiakeh, R., Alquran, M., Ali, M., Qureshi, S., Momani, S., Malkawi, A. A. R. (2024). Lie symmetry, convergence analysis, explicit solutions, and conservation laws for the timefractional modified Benjamin-Bona-Mahony equation. Journal of Applied Mathematics and Computational Mechanics, 23(1), 19-31.
- [6] T. Qawasmeh, H-Simulation functions and Ωb -distance mappings in the setting of Gb-metric spaces and application. Nonlinear Functional Analysis and Applications, 28 (2), 557-570, 2023. DOI: 10.22771/nfaa.2023.28.02.14
- [7] A. Bataihah, T. Qawasmeh, A New Type of Distance Spaces and Fixed Point Results, Journal of Mathematical Analysis, 15 (4), 81-90, 2024. doi.org/10.54379/jma-2024-4-5
- [8] W. Shatanawi, T. Qawasmeh, A. Bataihah, and A. Tallafha, "New Contractions and Some Fixed Point Results with Application Based on Extended Quasi b-Metric Spaces," U.P.B. Sci. Bull., Series A, Vol. 83, No. 2, 2021, pp. 1223-7027.
- [9] T. Qawasmeh, W. Shatanawi, A. Bataihah, and A. Tallafha, "Fixed Point Results and (α, β) - Triangular Admissibility in the Frame of Complete Extended b-Metric Spaces and Application," U.P.B. Sci. Bull., Series A, Vol. 83, No. 1, 2021, pp. 113-124.
- [10] Bataihah, A., Shatanawi, W., and Tallafha, A. (2020). FIXED POINT RESULTS WITH SIMULATION FUNCTIONS. Nonlinear Functional Analysis and Applications, 25(1), 13-23. DOI: 10.22771/nfaa.2020.25.01.02
- [11] Malkawi, A. A. R. M., Mahmoud, D., Rabaiah, A. M., Al-Deiakeh, R., and Shatanawi, W. (2024). ON FIXED POINT THEOREMS IN MR-METRIC SPACES. Nonlinear Functional Analysis and Applications, 1125-1136.
- [12] Gharib, G. M., Malkawi, A. A. R. M., Rabaiah, A. M., Shatanawi, W. A., Alsauodi, M. S. (2022). A Common Fixed Point Theorem in an M*-Metric Space and an Application. Nonlinear Functional Analysis and Applications, 27(2), 289-308.

- [13] Abodayeh, K., Shatanawi, W., Bataihah, A., and Ansari, A. H. (2017). Some fixed point and common fixed point results through Ω -distance under nonlinear contractions. *Gazi University Journal of Science*, 30(1), 293-302.
- [14] Bataihah, A., Tallafha, A., and Shatanawi, W. (2020). Fixed point results with Ω -distance by utilizing simulation functions. *Ital. J. Pure Appl. Math*, 43, 185-196.
- [15] K. Abodayeh, A. Bataihah, and W. Shatanawi, "Generalized Ω -Distance Mappings and Some Fixed Point Theorems," *UPB Sci. Bull. Ser. A*, Vol. 79, 2017, pp. 223-232.
- [16] Qawasmeh, T., Shatanawi, W., Bataihah, A., & Tallafha, A. (2019). Common Fixed Point Results for Rational $(\alpha, \beta)\phi$ - $m\phi$ Contractions in Complete Quasi Metric Spaces. *Mathematics*, 7(5), 392. DOI: 10.3390/math7050392
- [17] A. Rabaiah, A. Tallafha and W. Shatanawi, Common fixed point results for mappings under nonlinear contraction of cyclic form in b-Metric Spaces, *Advances in mathematics scientific journal*, 2021, 26(2), pp. 289–301.
- [18] Abu-Irwaq, I., Shatanawi, W., Bataihah, A., Nuseir, I. (2019). Fixed point results for nonlinear contractions with generalized Ω -distance mappings. *UPB Sci. Bull. Ser. A*, 81(1), 57-64.
- [19] Malkawi, A. A. R. M. (2025). Existence and Uniqueness of Fixed Points in MR-Metric Spaces and Their Applications. *EUROPEAN JOURNAL OF PURE AND APPLIED MATHEMATICS*, Vol. 18, No. 2, Article Number 6077.
- [20] Malkawi, A. A. R. M. (2025). Convergence and Fixed Points of Self-Mappings in MR-Metric Spaces: Theory and Applications. *EUROPEAN JOURNAL OF PURE AND APPLIED MATHEMATICS*, Vol. 18, No. 2, Article Number 5952.
- [21] Malkawi, A. A. R. M. (2025). Fixed Point Theorem in MR-metric Spaces VIA Integral Type Contraction. *WSEAS Transactions on Mathematics*, 24, 295-299.
- [22] Al-Sharif, S., and Malkawi, A. (2020). Modification of conformable fractional derivative with classical properties. *Ital. J. Pure Appl. Math*, 44, 30-39.
- [23] Gharib, G.M., Alsauodi, M.S., Guiatni, A., Al-Omari, M.A., Malkawi, A.A.-R.M., Using Atom Solution Method to Solve the Fractional Equations, *Springer Proceedings in Mathematics and Statistics*, 2023, 418, pp. 123–129.